WP2 Template 6
Simulation of an electrical oscillator supplying a resistor through a 4 diodes bridge full-wave rectifier

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1 Description of the physical problem : electrical oscillator with 4 diodes bridge full-wave rectifier

In this sample, a LC oscillator initialized with a given voltage across the capacitor and a null current through the inductor provides the energy to a load resistance through a full-wave rectifier consisting of a 4 ideal diodes bridge (see fig. 1).

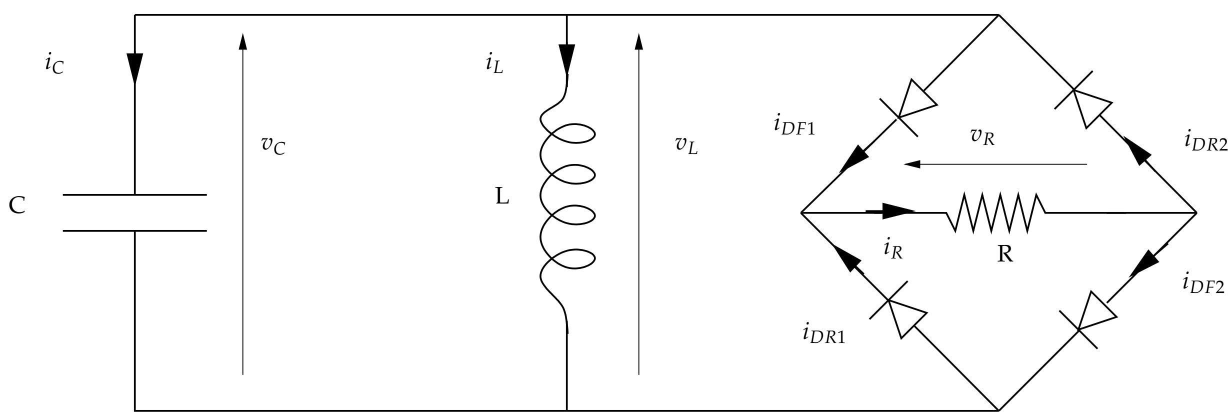

Figure 1: Electrical oscillator with 4 diodes bridge full-wave rectifier

Both waves of the oscillating voltage across the LC are provided to the resistor with current flowing always in the same direction. The energy is dissipated in the resistor resulting in a damped oscillation.

2 Definition of a general abstract class of NSDS : the linear time invariant complementarity system (LCS)

This type of non-smooth dynamical system consists of :

- a time invariant linear dynamical system (the oscillator). The state variable of this system is denoted by x.
- a non-smooth law describing the behaviour of each diode of the bridge as a complementarity condition between current and reverse voltage (variables (y, λ)). Depending on the diode position in the bridge, y stands for the reverse voltage across the diode or for the diode current.

- a linear time invariant relation between the state variable x and the non-smooth law variables (y, λ)

2.1 Dynamical system and Boundary conditions

Remark : In a more general setting, the system's evolution would be described by a DAE :

$$G \cdot x' = A \cdot x + E \cdot u + b + r$$

with G, A, E matrices constant over time (time invariant system), u, b source terms functions of time and r, a term coming from the non-smooth law variables : $r = B \cdot \lambda + a$ with B, a constant over time.
We will consider here the case of an ordinary differential equation :

$$x' = A \cdot x + E \cdot u + b + r$$

and an initial value problem for which the boundary conditions are $t_0 \in \mathbb{R}, x(t_0) = x_0$.

2.2 Relation between constrained variables and state variables

In the linear time invariant framework, the non-smooth law acts on the linear dynamical system evolution through the variable $r = B \cdot \lambda + a$. Reciprocally, the state variable x acts on the non-smooth law through the relation $y = C \cdot x + D \cdot \lambda + F \cdot u + e$ with C, D, F, e constant over time.

2.3 Definition of the Non Smooth Law between constrained variables

It is a complementarity condition between y and λ : $0 \leq y \perp \lambda \geq 0$. This corresponds to the behaviour of the rectifying diodes, as described in 3.3.

3 The formalization of the electrical oscillator with 4 diodes bridge full-wave rectifier into the LCS

The equations come from the following physical laws :

- the Kirchhoff current law (KCL) establishes that the sum of the currents arriving at a node is zero,
- the Kirchhoff voltage law (KVL) establishes that the sum of the voltage drops in a loop is zero,
- the branch constitutive equations define the relation between the current through a bipolar device and the voltage across it

Refering to figure 1, the Kirchhoff laws could be written as :

$$\begin{aligned}
&v_L = v_C \\
&v_L = v_{DF1} - v_{DR1} \\
&v_{DF1} + v_R + v_{DR2} = 0 \\
&v_{DF2} + v_R + v_{DR1} = 0 \\
&i_C + i_L + i_{DF1} - i_{DR2} = 0 \\
&i_{DF1} + i_{DR1} = i_R \\
&i_{DF2} + i_{DR2} = i_R
\end{aligned}$$

while the branch constitutive equations for linear devices are :

$$\begin{aligned}
&i_C = Cv'_C \\
&v_L = Li'_L \\
&v_R = Ri_R
\end{aligned}$$

and last the "branch constitutive equation" of the ideal diodes that is no more an equation but instead a complementarity condition :

$$\begin{array}{l} 0 \leq i_{DF1} \perp -v_{DF1} \geq 0 \\ 0 \leq i_{DR1} \perp -v_{DR1} \geq 0 \\ 0 \leq i_{DF2} \perp -v_{DF2} \geq 0 \\ 0 \leq i_{DR2} \perp -v_{DR2} \geq 0 \end{array}$$

This is illustrated on figure 2 where the left-hand sketch displays the ideal diode characteristic and the right-hand sketch displays the usual exponential characteristic as stated by Shockley's law.

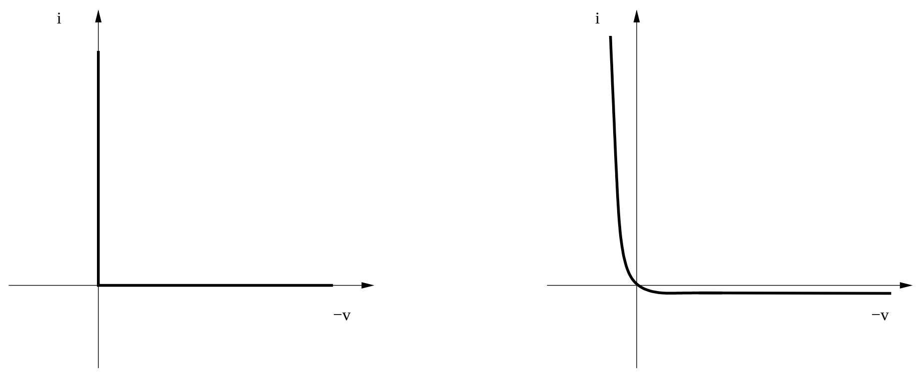

Figure 2: Non-smooth and smooth characteristics of a diode

3.1 Dynamical equation

After rearranging the previous equations, we obtain :

$$\begin{pmatrix} v'_L \\ i'_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{-1}{C} & \frac{1}{C} \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -v_{DR1} \\ -v_{DF2} \\ i_{DF1} \\ i_{DR2} \end{pmatrix}$$

that fits in the frame of 2 with

$$x = \begin{pmatrix} v_L \\ i_L \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} -v_{DR1} \\ -v_{DF2} \\ i_{DF1} \\ i_{DR2} \end{pmatrix}$$

3.2 Relations

We recall that the $r = B \cdot \lambda + a$ equation is expressed with

$$r = \begin{pmatrix} 0 & 0 & \frac{-1}{C} & \frac{1}{C} \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -v_{DR1} \\ -v_{DF2} \\ i_{DF1} \\ i_{DR2} \end{pmatrix}$$

from the dynamical equation (3.1).
Rearranging the initial set of equations yields :

$$\begin{pmatrix} i_{DR1} \\ i_{DF2} \\ -v_{DF1} \\ -v_{DR2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & \frac{1}{R} & -1 & 0 \\ \frac{1}{R} & \frac{1}{R} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -v_{DR1} \\ -v_{DF2} \\ i_{DF1} \\ i_{DR2} \end{pmatrix}$$

as the second equation of the linear time invariant relation with

$$y = \begin{pmatrix} i_{DR1} \\ i_{DF2} \\ -v_{DF1} \\ -v_{DR2} \end{pmatrix}$$

3.3 Non Smooth laws

There is just the complementarity condition resulting from the ideal diode characteristic :

$$\begin{array}{c} 0 \leq -v_{DR1} \perp i_{DR1} \geq 0 \\ 0 \leq -v_{DF2} \perp i_{DF2} \geq 0 \\ 0 \leq i_{DF1} \perp -v_{DF1} \geq 0 \\ 0 \leq i_{DR2} \perp -v_{DR2} \geq 0 \end{array}$$

4 Description of the numerical strategy: the Moreau's time-stepping scheme

4.1 Time discretization of the dynamical system

The integration of the ODE over a time step $[t_i, t_{i+1}]$ of length h is :

$$\int_{t_i}^{t_{i+1}} x' \, dt = \int_{t_i}^{t_{i+1}} A \cdot x \, dt + \int_{t_i}^{t_{i+1}} (E \cdot u + b) dt + \int_{t_i}^{t_{i+1}} r \, dt$$

The left-hand term is $x(t_{i+1}) - x(t_i)$.
Right-hand terms are approximated this way :

- $\int_{t_i}^{t_{i+1}} A \cdot x \, dt$ is approximated using a θ-method

$$\int_{t_i}^{t_{i+1}} A \cdot x \, dt \approx h\theta(A \cdot x(t_{i+1})) + h(1-\theta)(A \cdot x(t_i))$$

- since the second integral comes from independent sources, it can be evaluated with whatever quadrature method, for instance a θ-method

$$\int_{t_i}^{t_{i+1}} (E \cdot u + b) dt \approx h\theta(E \cdot u(t_{i+1}) + b(t_{i+1})) + h(1-\theta)(E \cdot u(t_i) + b(t_i))$$

- the third integral is approximated like in an implicit Euler integration

$$\int_{t_i}^{t_{i+1}} r \, dt \approx hr(t_{i+1})$$

By replacing the accurate solution $x(t_i)$ by the approximated value x_i, we get :

$$x_{i+1} - x_i = h\theta(A \cdot x_{i+1}) + h(1-\theta)(A \cdot x_i) + h\theta(E \cdot u(t_{i+1}) + b(t_{i+1})) + h(1-\theta)(E \cdot u(t_i) + b(t_i)) + hr_{i+1}$$

Assuming that $I - h\theta A$ is invertible, matrix W is defined as $(I - h\theta A)^{-1}$. We get then :

$$x_{i+1} = W(I + h(1-\theta)A) \cdot x_i + W(h\theta(E \cdot u(t_{i+1}) + b(t_{i+1})) + h(1-\theta)(E \cdot u(t_i) + b(t_i))) + hWr_{i+1}$$

An intermediate variable x_{free} related to the smooth part of the system is defined as :

$$x_{free} = W(I + h(1-\theta)A) \cdot x_i + W(h\theta(E \cdot u(t_{i+1}) + b(t_{i+1})) + h(1-\theta)(E \cdot u(t_i) + b(t_i)))$$

Thus the calculus of x_{i+1} becomes :

$$x_{i+1} = x_{free} + hWr_{i+1}$$

4.2 Time discretization of the relations

It comes straightforwardly :

$$r_{i+1} = B \cdot \lambda_{i+1} + a$$

$$y_{i+1} = C \cdot x_{i+1} + D \cdot \lambda_{i+1} + F \cdot u(t_{i+1}) + e$$

4.3 Time discretization of the non-smooth law

It comes straightforwardly :

$$0 \leq y_{i+1} \perp \lambda_{i+1} \geq 0$$

4.4 Summary of the time discretized equations

These equations are summarized assuming that there is no source term and simplified relations as for the electrical oscillator with full-wave rectifier.

$$\begin{array}{rcl}
W & = & (I - h\theta A)^{-1} \\
x_{free} & = & W(I + h(1-\theta)A) \cdot x_i \\
x_{i+1} & = & x_{free} + hWr_{i+1} \\
r_{i+1} & = & B \cdot \lambda_{i+1} \\
y_{i+1} & = & C \cdot x_{i+1} + D \cdot \lambda_{i+1} \\
\multicolumn{3}{c}{0 \leq y_{i+1} \perp \lambda_{i+1} \geq 0}
\end{array}$$

4.5 Numerical strategy

The integration algorithm with a fixed step is described here :

Algorithm 1 Integration of the electrical oscillator with 4 diodes bridge full-wave rectifier through a fixed Moreau time stepping scheme

Require: $R > 0, L > 0, C > 0$
Require: Time parameters h, T, t_0 and θ for the integration
Require: Initial value of inductor voltage $v_L = x_0(0)$
Require: Optional, initial value of inductor current $i_L = x_0(1)$ (default : 0)

$n_{step} = \frac{T-t_0}{h}$

//Dynamical system specification

$$A = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix}$$

//Relation specification

$$B = \begin{pmatrix} 0 & 0 & \frac{-1}{C} & \frac{1}{C} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{1}{R} & \frac{1}{R} & -1 & 0 \\ \frac{1}{R} & \frac{1}{R} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

//Construction of time independent operators

Require: $I - h\theta A$ invertible

$W = (I - h\theta A)^{-1}$

$M = D + hCWB$

//Non-smooth dynamical system integration

for $i = 0$ to $n_{step} - 1$ **do**

$$\begin{aligned} x_{free} &= W(I + h(1-\theta)A)x_i && \text{// Computation of } x_{free} \\ q &= C \cdot x_{free} && \text{// Formalization of the one step LCP} \\ (y_{i+1}, \lambda_{i+1}) &= \text{solveLCP}(M, q) && \text{// One step LCP solving} \\ x_{i+1} &= x_{free} + hWB\lambda_{i+1} && \text{// Computation of new state} \end{aligned}$$

end for

5 Comparison with numerical results coming from SPICE models and algorithms

We have used the SMASH simulator from Dolphin to perform a simulation of this circuit with a smooth model of the diode as given by Shockley's law , with a classical one step solver (Newton-Raphson) and the trapezoidal integrator.

5.1 Characteristic of the diode in the SPICE model

The figure (3) depicts the static $I(V)$ characteristic of two diodes with default SPICE parameters and two values for the emission coefficient N : 1.0 (standard diode) and 0.25 (stiff diode).

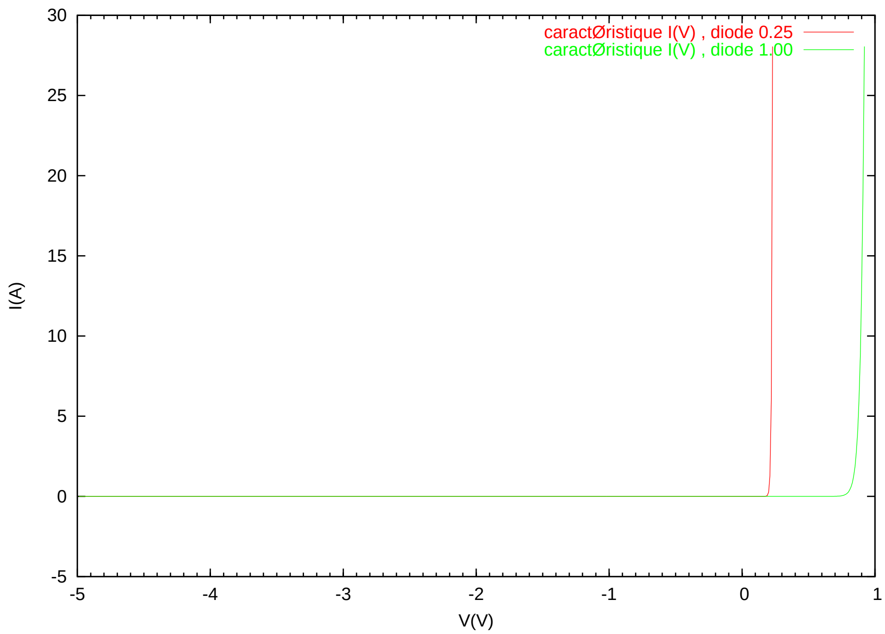

Figure 3: Diodes characteristics from SPICE model with $N = 0.25$ and $N = 1$

The stiff diode is close to an ideal one with a threshold of 0.2 V.

5.2 Simulation results

Figure (4) displays a comparison of the SMASH and SICONOS results with a trapezoidal integration ($\theta = 0.5$) and a fixed time step of 1 μs. A stiff diode model was used in SMASH simulations. One can notice that the results from both simulators are very close. The slight differences are due to the smooth model of the diode used by SMASH, and mainly to the threshold of around 0.2 V. Such a threshold yields small differences in the conduction state of the diode with respect to the ideal diode.

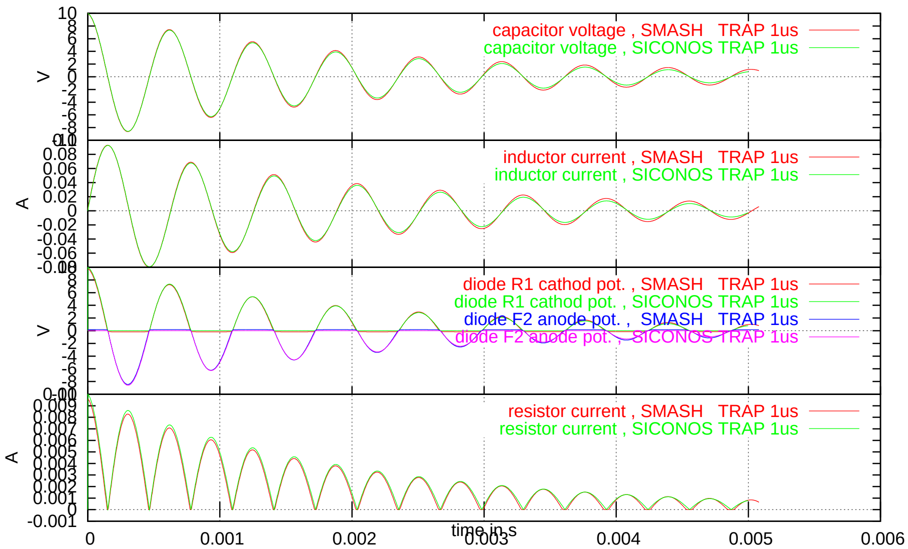

Figure 4: SMASH and SICONOS simulation results with trapezoidal integration, 1 μs time step